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A short time existence/uniqueness result for a nonlocal topology-preserving segmentation model

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ABSTRACT

Motivated by a prior applied work of Vese and the second author dedicated to segmentation under topological constraints, we derive a slightly modified model phrased as a functional minimization problem, and propose to study it from a theoretical viewpoint. The mathematical model leads to a second order nonlinear PDE with a singularity at $Du = 0$ and containing a nonlocal term. A suitable setting is thus the one of the viscosity solution theory and, in this framework, we establish a short time existence/uniqueness result as well as a Lipschitz regularity result for the solution.

1. Introduction

In [16], Le Guyader and Vese propose a topology-preserving segmentation model based on an implicit level-set formulation and on the geodesic active contours. The goal of this paper is to prove a short time existence and uniqueness result for a slightly modified model. The necessity of designing topology-preserving processes arises in medical imaging, for instance, in the human cortex reconstruction: it is well known that the human cortex has a spherical topology and this anatomical feature must be preserved through the segmentation process accordingly. The need for topology-preserving models also occurs when the shape to be detected must be homeomorphic to the initial one. To fix ideas, we propose some examples to illustrate what the results should be when running such a kind of algorithm. The implicit framework of the level-set method (see [17] for instance) has several advantages when tracking propagating fronts. In particular, it easily handles topological changes such as merging and breaking. Thus, in Fig. 1, when no topological contraints are enforced, the evolving contour splits into two components. On the other hand, in the topology-preserving framework (see Fig. 2),

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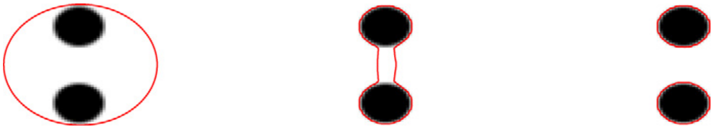

Fig. 1. Segmentation of the synthetic image with two disks when no topological constraint is enforced: the contour has split into two components. Iterations 0, 140, 180.

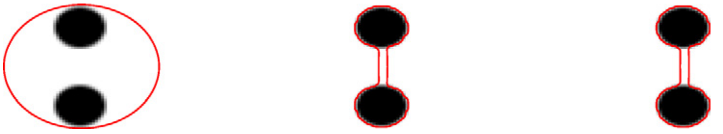

Fig. 2. Segmentation of the synthetic image with two disks when topological constraints are applied. Iterations 0, 180, 210.

we aim at segmenting the two disks while maintaining the same topology throughout the process, which means that we expect to get one path-connected component.

For the sake of completeness, we refer the reader to other prior related works dedicated to segmentation models under topological constraints: [1,8,13,16,18,19].

2. Modelling and exposition of the main result

2.1. Description of the model

The model proposed in [16] is as follows. Let Ω be a bounded open subset of $\mathbb{R}^2$, $\partial\Omega$ its boundary and let I be a given bounded image function defined by $I : \bar{\Omega} \to \mathbb{R}$. Let $\tilde{g} : [0, +\infty[\to [0, +\infty[$ be an edge-detector function satisfying $\tilde{g}(0) = 1$, $\tilde{g}$ strictly decreasing, and $\lim_{r\to+\infty} \tilde{g}(r) = 0$. The evolving contour $\mathcal{C}$ is embedded in a higher-dimensional Lipschitz continuous function Φ defined by $\Phi : \Omega \times [0, +\infty[\to \mathbb{R}$ with $(x,t) \mapsto \Phi(x,t)$ such that $\mathcal{C}(t,\cdot) = \{x \in \Omega \mid \Phi(x,t) = 0\}$ and

$$\begin{cases} \Phi < 0 & \text{on } w \text{ the interior of } \mathcal{C}, \\ \Phi > 0 & \text{on } \Omega \setminus \bar{w}. \end{cases}$$

Generally, this function Φ is preferred to be a signed-distance function for the stability of numerical computations.

The segmentation model of [16] combines the classical geodesic active contour functional (see [7]) with a topological constraint phrased in terms of a double integral. More precisely, it consists in minimizing the following functional:

$$F(\Phi) + \mu E(\Phi),$$

where $\mu > 0$ is a tuning parameter. The functional F stems from the geodesic active contour model and is defined by:

$$F(\Phi) = \int_{\Omega} \tilde{g}\big(\big|DI(x)\big|\big)\delta\big(\Phi(x)\big)\big|D\Phi(x)\big|\,dx,$$

with δ the 1-D Dirac measure. The functional E, related to the topological constraint, is defined by:

$$E(\Phi) = -\int_{\Omega}\int_{\Omega}\bigg[\exp\bigg(-\frac{\|x-y\|_2^2}{d^2}\bigg)\big\langle D\Phi(x), D\Phi(y)\big\rangle H\big(\Phi(x)+l\big)H\big(l-\Phi(x)\big)$$
$$\times H\big(\Phi(y)+l\big)H\big(l-\Phi(y)\big)\bigg]\,dx\,dy,$$

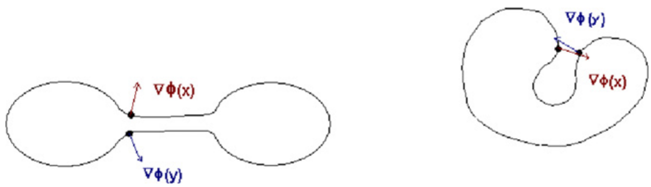

Fig. 3. Geometrical characterization of points in a zone where the curve is to split, merge or have a contact point.

with H the 1-D Heaviside function, $\langle\cdot,\cdot\rangle$ denoting the Euclidean scalar product in $\mathbb{R}^2$ and $\|\cdot\|_2$ the associated norm. A geometrical observation motivates the introduction of E. Indeed, in the case where Φ is a signed-distance function, $|D\Phi| = 1$ and the unit outward normal vector to the zero level line at point x is $D\Phi(x)$. Let us now consider two points $(x, y) \in \Omega \times \Omega$ belonging to the zero level line of Φ, close enough to each other, and let $D\Phi(x)$ and $D\Phi(y)$ be the two unit outward normal vectors to the contour at these points. As shown in Fig. 3, when the contour is about to merge or split, that is, when the topology of the evolving contour is to change, then $\langle D\Phi(x), D\Phi(y)\rangle \simeq -1$. This remark justifies the construction of E. Also, instead of working with only the points of the zero level line, the authors propose to focus on the points contained in a narrow band around the zero level line, more precisely, on the set of points $\{x \in \Omega \mid -l \leqslant \Phi(x) \leqslant l\}$, l being a level parameter. Lastly, the function $(x, y) \mapsto \exp(-\frac{\|x-y\|_2^2}{d^2})$ measures the nearness of the two points x and y. Thus if the unit outward normal vectors to the level lines passing through x and y have opposite directions, the functional is not minimal.

The Euler–Lagrange equation is derived and is solved by a gradient descent method. A rescaling is made by replacing $\delta(\Phi)$ by $|D\Phi|$ and the evolution equation is complemented by Neumann homogeneous boundary conditions. It leads to the following evolution problem:

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} = |D\Phi|\left[\operatorname{div}\left(\tilde{g}\big(|DI|\big)\dfrac{D\Phi}{|D\Phi|}\right)\right] + 4\dfrac{\mu}{d^2} H\big(\Phi(x)+l\big)H\big(l-\Phi(x)\big) \\ \qquad\qquad \times \displaystyle\int_\Omega \big[\big\langle x-y, D\Phi(y)\big\rangle e^{-\|x-y\|_2^2/d^2} H\big(\Phi(y)+l\big)H\big(l-\Phi(y)\big)\big]\,dy, \\ \Phi(x,0) = \Phi_0(x), \\ \dfrac{\partial \Phi}{\partial \vec{\nu}} = 0, \quad \text{on } \partial\Omega. \end{cases}$$

This problem is hard to handle from a theoretical point of view. A suitable setting would be the one of the viscosity solution theory (due to the nonlinearity induced by the modified mean curvature term) but the dependency of the nonlocal term to the gradient ($D\Phi(y)$) and the failure to fulfill the monotony property in Φ make it impossible. For this reason, we consider a slightly modified problem. We propose to focus on the following minimization problem for which the topological constraint is only applied to the zero level line (we still assume that $|D\Phi| = 1$),

$$\inf_\Phi \int_\Omega \tilde{g}\big(\big|DI(x)\big|\big)\delta\big(\Phi(x)\big)\big|D\Phi(x)\big|\,dx \\ - \mu\int_\Omega\int_\Omega \left[\exp\left(\frac{-\|x-y\|_2^2}{d^2}\right)\big\langle D\Phi(x), D\Phi(y)\big\rangle \delta\big(\Phi(x)\big)\delta\big(\Phi(y)\big)\right]dx\,dy.$$

We compute the Euler–Lagrange equation and apply a gradient descent method. We get the following evolution equation:

$$\begin{aligned}
\frac{\partial \Phi}{\partial t} &= \delta(\Phi)\operatorname{div}\left(\tilde{g}\big(|DI|\big)\frac{D\Phi}{|D\Phi|}\right) \\
&\quad - 2\mu\int_{\Omega}\frac{\partial}{\partial x_1}\left[\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)\right]\delta\big(\Phi(x)\big)\delta\big(\Phi(y)\big)\frac{\partial \Phi}{\partial y_1}(y)\,dy \\
&\quad - 2\mu\int_{\Omega}\frac{\partial}{\partial x_2}\left[\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)\right]\delta\big(\Phi(x)\big)\delta\big(\Phi(y)\big)\frac{\partial \Phi}{\partial y_2}(y)\,dy \\
&= \delta(\Phi)\left\{\operatorname{div}\left(\tilde{g}\big(|DI|\big)\frac{D\Phi}{|D\Phi|}\right)\right. \\
&\quad + \frac{4\mu}{d^2}\int_{\Omega}(x_1-y_1)\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)\frac{\partial}{\partial y_1}\big[H\big(\Phi(y)\big)\big]\,dy \\
&\quad \left.+ \frac{4\mu}{d^2}\int_{\Omega}(x_2-y_2)\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)\frac{\partial}{\partial y_2}\big[H\big(\Phi(y)\big)\big]\,dy\right\}.
\end{aligned}$$

Doing an integration by parts in the second part of the PDE and setting the necessary boundary conditions to zero, it yields:

$$\begin{aligned}
\frac{\partial \Phi}{\partial t} &= \delta(\Phi)\left\{\operatorname{div}\left(\tilde{g}\big(|DI|\big)\frac{D\Phi}{|D\Phi|}\right)\right. \\
&\quad \left.+ \frac{4\mu}{d^2}\int_{\Omega}\left(2-\frac{2}{d^2}\|x-y\|_2^2\right)\exp\left(-\frac{\|x-y\|_2^2}{d^2}\right)H\big(\Phi(y)\big)\,dy\right\} \\
&= \delta(\Phi)\left\{\operatorname{div}\left(\tilde{g}\big(|DI|\big)\frac{D\Phi}{|D\Phi|}\right) + c_0 * \big[\Phi(\cdot,t)\big]\right\},
\end{aligned}$$

with $[\Phi(\cdot,t)]$ the characteristic function of the set $\{\Phi(\cdot,t) > 0\}$ and

$$c_0 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}, \\ x \mapsto \frac{4\mu}{d^2}\left(2-\frac{2}{d^2}\|x\|_2^2\right)\exp\left(-\frac{\|x\|_2^2}{d^2}\right). \end{cases} \tag{2.1}$$

A rescaling can be made by replacing $\delta(\Phi)$ by $|D\Phi|$ in order to apply the same motion to all level sets. Also, for the sake of simplicity, we assume, in the sequel, that the problem is formulated on $\mathbb{R}^2$ for the spatial coordinates and we set $g(x) = \tilde{g}(DI(x))$.

Remark 1. This simplified model qualitatively performs in a similar way to [16], as shown in the following illustrations. The first example in Fig. 4 was taken from [13]: the two middle fingers touch so with the classical geodesic active contours, the evolving contour is going to merge and a hole will appear, which is undesirable. With the proposed model, the repelling forces prevent the curve from merging. The method has also been tested on complex slices of the brain (Courtesy of Laboratory Of NeuroImaging, UCLA). We can see (Fig. 5) that the method enables us to get the details of the brain envelope without creating contact points. This is complicated in this example since the slice shows two disconnected parts that are very close to each other.

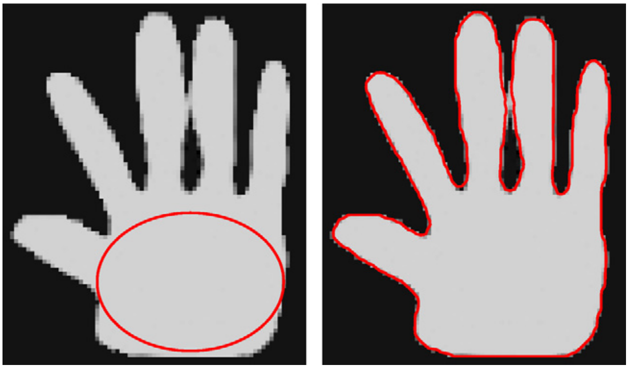

Fig. 4. Segmentation of the hand image taken from [13] with the proposed topology-constrained method.

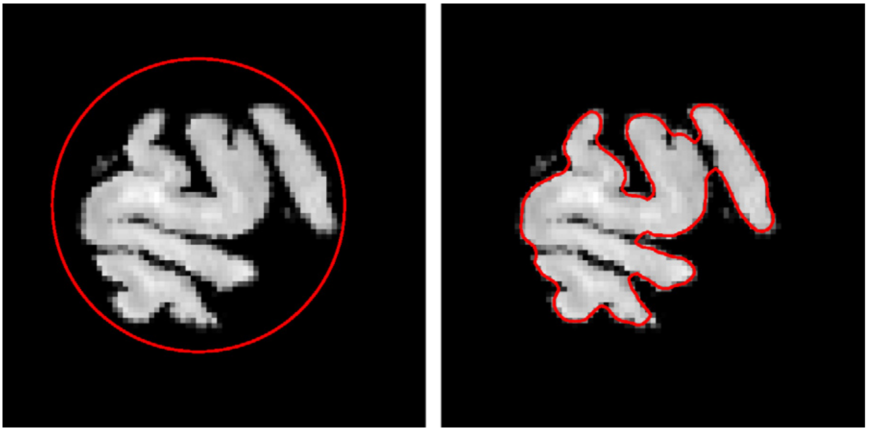

Fig. 5. Segmentation of a slice of the brain with topological constraints.

2.2. Main results

The goal of the paper is to provide a result of existence/uniqueness for the nonlocal topology-preserving segmentation model depicted above. The result is stated as follows.

Given $T > 0$, we consider the following problem: find $u(x,t)$ solution of:

$$\begin{cases} \dfrac{\partial u}{\partial t} = |Du| \left\{ \operatorname{div}\left(g \dfrac{Du}{|Du|} \right) + c_0 * \left[u(\cdot, t) \right] \right\} & \text{in } \mathbb{R}^2 \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{2.2}$$

with u_0 such that $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$ (we denote by B_0 its Lipschitz constant). In particular, u_0 is $L^{\infty}_{loc}(\mathbb{R}^2)$. We need the following assumptions on function g:

(H1) $\exists \delta > 0$, $\forall x \in \mathbb{R}^2$, $\delta < g(x) \leqslant 1$.

(H2) g, $g^{\frac{1}{2}}$ and Dg are bounded and Lipschitz continuous on $\mathbb{R}^2$ with Lipschitz constant κ_g, $\kappa_{g^{\frac{1}{2}}}$ and κ_{Dg} respectively. For simplicity of notation, we set $L_g = \max(\kappa_g, \kappa_{Dg}, \kappa_{g^{\frac{1}{2}}})$.

Remark 2. Owing to the hypothesis made on function g in Subsection 2.1, assumption (H1) just means that the gradient of I is bounded. In practice, this is the case since for a grey-level image, the intensity of a pixel is either an integer between 0 and 255 or a real number between 0 and 1. Assumption (H2) is rather classical.

This model is a nonlocal Hamilton–Jacobi equation. We propose, in this paper, to prove a short time existence and uniqueness result for this equation.

Theorem 1 *(Short time existence and uniqueness). Assume* (H1)–(H2) *and let* $u_0 : \mathbb{R}^2 \to \mathbb{R}$ *be such that* $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$ *and*:

$$|Du_0| < B_0 \quad \textit{in } \mathbb{R}^2 \quad \textit{and} \quad \frac{\partial u_0}{\partial x_2} > b_0 > 0 \quad \textit{in } \mathbb{R}^2.$$

Let c_0 *be defined in* (2.1)*. Then there exists* $T^* > 0$ (*depending only on* b_0*,* B_0*,* c_0 *and* g) *such that there exists a unique viscosity solution of problem* (2.2) *in* $\mathbb{R}^2 \times [0, T^*)$*. Moreover, the solution is Lipschitz continuous in space and time.*

Since the equation is nonlinear, as previously mentioned, a natural framework is the one of the viscosity solution theory introduced by Crandall and Lions [9] (see for instance the monographs of Barles [5] and Bardi and Capuzzo-Dolcetta [4] for a presentation of first order equations and the papers of Crandall, Ishii and Lions [10] and Barles [6] for the second order case). Our work is much motivated by a previous article of the first author ([11]), which is dedicated to the mathematical study of a model for dislocation dynamics with a mean curvature term. The main difference with the model in [11] is that in our case, the PDE explicitly depends on the space variable x, which induces substential adaptations of the proof. The strategy of the proof is the same as the one applied in [3] or [11], *i.e.*, using a fixed point method by freezing the nonlocal term. More precisely, we apply a fixed point method on a functional space E (defined later on) whose definition lies, in particular, in estimations on the gradients. The key point is thus to get estimates on the Lipschitz constant in space and time of the solution as well as a bound from below on the gradient in space. The main difficulties come from the fact that the mean curvature term is balanced by a function of the space variable x and so, to obtain the estimate from below on the gradient, we have to bound the mean curvature term. This is done using the Lipschitz regularity of the solution.

The outline of the paper is as follows. Section 3 is devoted to the mathematical study of a related preliminary local problem, which is useful to establish the existence/uniqueness of the solution of the nonlocal problem. We give an existence/uniqueness result for the solution of the local problem and provide some results on the regularity of this solution. Section 4 presents the main result of the paper, that is, a short time existence/uniqueness result for the nonlocal problem.

3. Study of a related local problem

As aforementioned, we start by looking into a related local problem. This study will enable us to establish the analytical results for the nonlocal problem.

Given $T > 0$, we consider the following problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = c(x,t)|Du| + |Du|\,\mathrm{div}\left(g(x)\dfrac{Du}{|Du|}\right) & \text{on } \mathbb{R}^2 \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2, \end{cases} \tag{3.3}$$

with $c : \mathbb{R}^2 \times [0,T) \mapsto c(x,t)$ bounded, Lipschitz continuous in space (we denote by L_c its Lipschitz constant in space), and in time (we denote by L_{c_t} its Lipschitz constant in time).

The evolution equation can be rewritten in the form

$$\frac{\partial u}{\partial t} + G\big(x, t, Du, D^2u\big) = 0,$$

with $G : \mathbb{R}^2 \times [0, T) \times \mathbb{R}^2 \times \mathcal{S}^2$ ($\mathcal{S}^2$ being the set of symmetric 2×2 matrices equipped with its natural partial order) defined by:

$$\begin{aligned} G(x, t, p, X) &= -c(x, t)|p| + F(x, p, X) \\ &= -c(x, t)|p| + g(x)H(p, X) - \big\langle \nabla g(x), p \big\rangle, \end{aligned}$$

with the following properties:

1. The operators G, F and $H : (p, X) \mapsto -\mathrm{trace}((I - \frac{p \otimes p}{|p|^2})X)$ are independent of u and are elliptic, *i.e.*, $\forall X, Y \in \mathcal{S}^2$, $\forall p \in \mathbb{R}^2$,

$$\text{if } X \leqslant Y \quad \text{then } F(x, p, X) \geqslant F(x, p, Y).$$

2. F is locally bounded on $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}^2$, continuous on $\mathbb{R}^2 \times \mathbb{R}^2 - \{0_{\mathbb{R}^2}\} \times \mathcal{S}^2$, and $F^*(x, 0, 0) = F_*(x, 0, 0) = 0$, where F^* (resp. F_*) is the upper semi-continuous (usc) envelope (resp. lower semi-continuous (lsc) envelope) of F.

3.1. Existence and uniqueness

The first important result is a comparison principle that states that if a sub-solution and a super-solution are ordered at initial time then they are ordered at any time. We refer the reader to [10] for the definition of viscosity solutions.

Theorem 2 *(Comparison principle). Assume* (H1)–(H2) *and let* $u : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$ *be a locally bounded and upper semi-continuous sub-solution and* $v : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$ *be a locally bounded and lower semi-continuous supersolution of* (3.3)*. Assume that* $u(x, 0) \leqslant u_0(x) \leqslant v(x, 0)$ *in* $\mathbb{R}^2$*, then* $u \leqslant v$ *in* $\mathbb{R}^2 \times [0, T)$.

Proof. This proof is rather classical. For the reader's convenience, we refer to [12], in which the authors prove comparison theorems for viscosity solutions of related degenerate parabolic equations of general form in a domain not necessarily bounded. □

We now turn to the existence of a solution. In this prospect, we use the classical Perron's method [14] and need to construct barriers.

Proposition 1 *(Existence of barriers). Assume* (H1)–(H2) *and let* u_0 *be such that* $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$*. Then there exists a constant* $C_1 > 0$ *depending only on* $\|c\|_{L^\infty}$*,* g *and* u_0 *such that* $u^\pm = u_0 \pm C_1 t$ *are respectively super- and sub-solution of* (3.3).

Proof. Let us check that u^+ is a supersolution (the proof for u^- being similar). (This is written in a formal way but this is easy to show using test functions). We have:

$$\begin{aligned} c(x, t)\big|Du^+\big| - F^*\big(x, Du^+, D^2u^+\big) &= c(x, t)|Du_0| - g(x)H^*\big(Du_0, D^2u_0\big) + \big\langle Dg(x), Du_0\big\rangle \\ &\leqslant \|c\|_{L^\infty}\|Du_0\|_{L^\infty} + L_g\|Du_0\|_{L^\infty} + \sup_{x \in \mathbb{R}^2}\big(-g(x)H^*\big(Du_0, D^2u_0\big)\big) \\ &\leqslant C_1 = \big(u^+\big)_t, \end{aligned}$$

if we choose

$$C_1 \geqslant \|c\|_{L^\infty}\|Du_0\|_{L^\infty} + \sup_{x\in\mathbb{R}^2}\bigl(-g(x)H^*\bigl(Du_0, D^2u_0\bigr)\bigr) + L_g\|Du_0\|_{L^\infty}.$$

Note that the supremum of $-g(x)H^*(Du_0, D^2u_0)$ is indeed bounded: H^* is upper semi-continuous, H_* is lower semi-continuous (so $-H_*$ is upper semi-continuous) and $H_* \leqslant H^*$. The function g being positive, we have $-g(x)H^*(Du_0, D^2u_0) \leqslant -g(x)H_*(Du_0, D^2u_0)$. As $-g(x)H_*(Du_0, D^2u_0)$ is upper semi-continuous and Du_0, D^2u_0 are bounded as well as g, it follows that $-g(x)H_*(Du_0, D^2u_0)$ is bounded from above and so is $-g(x)H^*(Du_0, D^2u_0)$. □

A direct consequence of the two previous results is the following existence/uniqueness theorem.

Theorem 3 *(Existence/uniqueness). Assume* (H1)–(H2) *and that u_0 is such that $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$. Then there exists a unique continuous solution of* (3.3) *on $\mathbb{R}^2 \times [0, T)$. Moreover, the solution satisfies for $(x, t) \in \mathbb{R}^2 \times [0, T)$,*

$$u_0(x) - C_1 t \leqslant u(x, t) \leqslant u_0(x) + C_1 t,$$

where C_1 is defined in Proposition 1.

Proof. This is a direct application of Perron's method (see [14]) joint to the comparison principle (Theorem 2). □

3.2. Regularity results

We now prove that the solution of problem (3.3) is Lipschitz continuous in space and time, and derive a lower bound on the partial derivative $\frac{\partial u}{\partial x_2}$. As previously mentioned, these bounds are required to apply the fixed point method in the space E.

Theorem 4 *(Lipschitz regularity in space). Assume* (H1)–(H2) *and that $\|Du_0\|_{L^\infty(\mathbb{R}^2)} \leqslant B_0$ with $B_0 > 0$. Then the solution of* (3.3) *is Lipschitz continuous in space and satisfies*:

$$\bigl\|Du(\cdot, t)\bigr\|_{L^\infty(\mathbb{R}^2)} \leqslant B(t),$$

with $B(t) = e^{C_2 t}B_0$ and $C_2 = L_c + L_g + 5L_g^2$.

Proof. The proof is similar to the one of Lemma 4.15 in [11], except that an additional difficulty emerges from the dependency in x of the modified mean curvature component. In particular, the third step of the proof which consists in establishing the viscosity inequalities by means of Theorem 8.3 of [10] is more complex. We refer the reader to [15] to see how the issue related to this dependency in x can be addressed for local problems. □

We now prove that the solution is Lipschitz continuous in time and estimate the associated Lipschitz constant.

Proposition 2 *(Lipschitz regularity in time). Let u_0 be such that $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$. Then the solution u of* (3.3) *is Lipschitz continuous in time and satisfies*:

$$\bigl\|u_t(x, \cdot)\bigr\|_{L^\infty(0,T)} \leqslant C_1 + L_{c_t}\int_0^T B(s)\,ds,$$

where C_1 is defined in Proposition 1.

Proof. We recall, from Theorem 3, that

$$\left|u(x,t)-u_0(x)\right| \leqslant C_1 t.$$

Let $h>0$ be such that $t+h \leqslant T$. We denote by

$$M=\sup_{x\in\mathbb{R}^2}\left|u(x,h)-u_0(x)\right| \leqslant C_1 h$$

and

$$u_h(x,t)=u(x,t+h)-L_{c_t}h\int_0^{t+h}B(s)\,ds-M.$$

Then u_h is still a sub-solution of (3.3). Indeed, formally, we have

$$\begin{aligned}(u_h)_t(x,t)&=u_t(x,t+h)-L_{c_t}hB(t+h)\\&=c(x,t+h)\left|Du(x,t+h)\right|-F\left(x,Du(x,t+h),D^2u(x,t+h)\right)-L_{c_t}hB(t+h)\\&\leqslant c(x,t)\left|Du_h(x,t)\right|-F\left(x,Du_h(x,t),D^2u_h(x,t)\right).\end{aligned}$$

Hence, using the comparison principle, one has $u_h(x,t)\leqslant u(x,t)$, that is

$$u(x,t+h)-u(x,t)\leqslant M+L_{c_t}h\int_0^{t+h}B(s)\,ds\leqslant C_1h+L_{c_t}h\int_0^{T}B(s)\,ds.$$

Similarly, one obtains that

$$\left|u(x,t+h)-u(x,t)\right|\leqslant C_1h+L_{c_t}h\int_0^{T}B(s)\,ds.$$

In conclusion, u is Lipschitz continuous in time with Lipschitz constant equal to $C_1+L_{c_t}\int_0^T B(s)\,ds$. □

We now turn to the prescribing of a lower bound on the gradient. We need the following lemma:

Lemma 1 *(Estimate on the curvature). Let $(p,Y)\in\mathbb{R}^2\times\mathcal{S}^2$ such that $\exists\tau\in\mathbb{R}$ such that $(\tau,p,Y)\in\bar{\mathcal{P}}^-u(y,t)$ (respectively $(\tau,p,Y)\in\bar{\mathcal{P}}^+u(y,t)$) ($u$ being a viscosity solution of the problem, it is also a viscosity sub- and super-solution). Then*

$$\begin{aligned}&-H^*(p,Y)\leqslant\frac{C_1+L_{c_t}TB(T)+\|c\|_{L^\infty(\mathbb{R}^2\times[0,T))}B(T)+L_gB(T)}{\delta}=:C_3\\&\left(\text{resp. } H_*(p,Y)\leqslant C_3\right),\end{aligned}$$

where C_1 denotes the Lipschitz constant in time of u and $B(\cdot)$ is defined in Theorem 4.

Proof. We only do the proof for $(\tau, p, Y) \in \bar{\mathcal{P}}^- u(y,t)$, the other one being similar. By definition,

$$\tau - c(y,t)|p| + g(y)H^*(p,Y) - \langle Dg(y), p\rangle \geqslant 0.$$

That is,

$$g(y)H^*(p,Y) \geqslant -\tau + c(y,t)|p| + \langle Dg(y), p\rangle.$$

But $-\tau \geqslant -C_1 - L_{c_t} T B(T)$ and $|p| \leqslant B(t) \leqslant B(T)$. Consequently,

$$g(y)H^*(p,Y) \geqslant -C_1 - L_{c_t} T B(T) - \|c\|_{L^\infty(\mathbb{R}^2\times[0,T))} B(T) - L_g B(T),$$

and

$$-H^*(p,Y) \leqslant \frac{C_1 + L_{c_t} T B(T) + \|c\|_{L^\infty(\mathbb{R}^2\times[0,T))} B(T) + L_g B(T)}{\delta}. \qquad \square$$

Theorem 5 *(Lower bound on the gradient). Let u_0 be such that $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$ and $\frac{\partial u_0}{\partial x_2} \geqslant b_0$ with $b_0 > 0$. Then the solution of* (3.3) *satisfies*:

$$\frac{\partial u}{\partial x_2} \geqslant b(t),$$

with $b(t) = b_0 - 2(L_c + L_g)\frac{B_0}{C_2}(e^{C_2 t} - 1) - (L_c + L_g + L_g C_3)t$, where C_2 and C_3 are defined respectively in Theorem 4 *and Lemma* 1.

Proof. We set $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We aim to prove that for $x_2 < y_2$, $u(x_1, y_2, t) - u(x_1, x_2, t) \geqslant b(t)(y_2 - x_2)$. In this purpose, let us introduce

$$M = \sup_{(x_1,x_2,y_2,t)|x_2<y_2} \big\{u(x_1,x_2,t) - u(x_1,y_2,t) - b(t)(x_2 - y_2)\big\},$$

and let us prove that $M \leqslant 0$.

We argue by contradiction. Let us assume that $M > 0$. Note that this supremum is bounded above. Indeed, from Theorem 3,

$$u_0(x) - C_1 t \leqslant u(x,t) \leqslant u_0(x) + C_1 t,$$

so

$$\begin{aligned} u_0(x_1,x_2) - u_0(x_1,y_2) - 2C_1 t - b(t)(x_2 - y_2) &\leqslant u(x_1,x_2,t) - u(x_1,y_2,t) - b(t)(x_2 - y_2) \\ &\leqslant u_0(x_1,x_2) - u_0(x_1,y_2) + 2C_1 t - b(t)(x_2 - y_2). \end{aligned}$$

But from the hypotheses $u_0(x_1,y_2) - u_0(x_1,x_2) \geqslant b_0(y_2 - x_2) \geqslant b(t)(y_2 - x_2)$ (as b is decreasing), so $u_0(x_1,x_2) - u_0(x_1,y_2) - b(t)(x_2 - y_2) \leqslant 0$ and $u(x_1,x_2,t) - u(x_1,y_2,t) - b(t)(x_2 - y_2) \leqslant 2C_1 T$. The first step of the proof consists in introducing a penalization and duplicating the variables as follows. We set:

$$\bar{M} = \sup_{(x_1,x_2,y_1,y_2,t)|x_2<y_2} \left\{ u(x_1,x_2,t) - u(y_1,y_2,t) - b(t)(x_2-y_2) - \frac{|x_1-y_1|^2}{2\epsilon} \right.$$
$$\left. - \frac{\gamma}{T-t} - \frac{\alpha}{2}\left(|x|^2+|y|^2\right) \right\}.$$

For α and γ small enough, $\bar{M} \geqslant \frac{M}{2} > 0$. Moreover, thanks to the term $-\frac{\alpha}{2}(|x|^2+|y|^2)$, this supremum is reached in $(\bar{x_1}, \bar{x_2}, \bar{y_1}, \bar{y_2}, \bar{t})$.

The second step of the proof consists in proving that $\bar{t} \neq 0$. By contradiction, let us assume that $\bar{t} = 0$. We then have:

$$0 < \frac{M}{2} \leqslant \bar{M} \leqslant u_0(\bar{x_1}, \bar{x_2}) - u_0(\bar{y_1}, \bar{y_2}) - b_0(\bar{x_2} - \bar{y_2}) - \frac{|\bar{x_1} - \bar{y_1}|^2}{2\epsilon}$$
$$\leqslant B_0|\bar{x_1} - \bar{y_1}| - \frac{|\bar{x_1} - \bar{y_1}|^2}{2\epsilon} + u_0(\bar{y_1}, \bar{x_2}) - u_0(\bar{y_1}, \bar{y_2}) - b_0(\bar{x_2} - \bar{y_2}).$$

A study of the function $h : r \mapsto B_0 r - \frac{r^2}{2\epsilon}$ on $\mathbb{R}^+$ gives us that it is bounded above by $\frac{B_0^2\epsilon}{2}$. Thus,

$$0 < \frac{M}{2} \leqslant \bar{M} \leqslant \frac{B_0^2\epsilon}{2} + u_0(\bar{y_1}, \bar{x_2}) - u_0(\bar{y_1}, \bar{y_2}) - b_0(\bar{x_2} - \bar{y_2}),$$

where we have that $u_0(\bar{y_1}, \bar{x_2}) - u_0(\bar{y_1}, \bar{y_2}) - b_0(\bar{x_2} - \bar{y_2}) \leqslant 0$ according to the assumptions on u_0. We clearly raise a contradiction for ϵ small enough, so $\bar{t} \neq 0$.

The third step of the proof consists in proving that $\bar{x_2} \neq \bar{y_2}$. We have:

$$0 < \frac{M}{2} \leqslant \bar{M} = u(\bar{x_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{x_2}, \bar{t}) + u(\bar{y_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{y_2}, \bar{t}) - b(\bar{t})(\bar{x_2} - \bar{y_2})$$
$$- \frac{|\bar{x_1} - \bar{y_1}|^2}{2\epsilon} - \frac{\gamma}{T-\bar{t}} - \frac{\alpha}{2}\left(|\bar{x}|^2 + |\bar{y}|^2\right)$$
$$\leqslant B(\bar{t})|\bar{x_1} - \bar{y_1}| - \frac{|\bar{x_1} - \bar{y_1}|^2}{2\epsilon} + u(\bar{y_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{y_2}, \bar{t}) - b(\bar{t})(\bar{x_2} - \bar{y_2})$$
$$\leqslant \frac{B(\bar{t})^2\epsilon}{2} + u(\bar{y_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{y_2}, \bar{t}) - b(\bar{t})(\bar{x_2} - \bar{y_2}).$$

Thus, for ϵ small enough,

$$u(\bar{y_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{y_2}, \bar{t}) - b(\bar{t})(\bar{x_2} - \bar{y_2}) \geqslant \frac{M}{3}.$$

Consequently, $\bar{x_2} \neq \bar{y_2}$.

The last step of the proof consists in raising a contradiction ensuring that $M \leqslant 0$. We consider $\Phi(x,y,t) = b(t)(x_2 - y_2) + \frac{|x_1-y_1|^2}{2\epsilon} + \frac{\gamma}{T-t}$ and we set $\bar{p} = \bar{x_1} - \bar{y_1}$. We use the parabolic version of Ishii's lemma [10] and we set:

$$\begin{cases} p_1 = D_x\Phi(\bar{x},\bar{y},\bar{t}) = p_2 = -D_y\Phi(\bar{x},\bar{y},\bar{t}) = \begin{pmatrix} \epsilon^{-1}\bar{p} \\ b(\bar{t}) \end{pmatrix} \neq 0 \quad \text{for } T \text{ small enough}, \\ A = D^2\Phi(\bar{x},\bar{y},\bar{t}) = \begin{pmatrix} \frac{1}{\epsilon} & 0 & -\frac{1}{\epsilon} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\epsilon} & 0 & \frac{1}{\epsilon} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{cases}$$

Then for all $\eta > 0$, there exist X and Y such that:

$$\begin{cases} \tau_1 - \tau_2 = b'(\bar{t})(\bar{x_2} - \bar{y_2}) + \dfrac{\gamma}{(T-\bar{t})^2}, \\ (\tau_1, p_1 + \alpha\bar{x}, X + \alpha I) \in \bar{\mathcal{P}}^+ u(\bar{x},\bar{t}), \\ (\tau_2, p_1 - \alpha\bar{y}, Y - \alpha I) \in \bar{\mathcal{P}}^- u(\bar{y},\bar{t}), \\ -\left(\dfrac{1}{\eta} + \|A\|\right) I \leqslant \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leqslant A + \eta A^2 = \begin{pmatrix} \frac{1}{\epsilon} + \frac{2\eta}{\epsilon^2} & 0 & -\frac{1}{\epsilon} - \frac{2\eta}{\epsilon^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\epsilon} - \frac{2\eta}{\epsilon^2} & 0 & \frac{1}{\epsilon} + \frac{2\eta}{\epsilon^2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{cases}$$

Because u is a solution,

$$\tau_1 - c(\bar{x},\bar{t})|p_1 + \alpha\bar{x}| + F_*(\bar{x}, p_1 + \alpha\bar{x}, X + \alpha I) \leqslant 0,$$
$$\tau_2 - c(\bar{y},\bar{t})|p_1 - \alpha\bar{y}| + F^*(\bar{y}, p_1 - \alpha\bar{y}, Y - \alpha I) \geqslant 0.$$

Then, substracting the two previous inequalities yields:

$$\begin{aligned} b'(\bar{t})(\bar{x_2} - \bar{y_2}) + \frac{\gamma}{T^2} &\leqslant c(\bar{x},\bar{t})|p_1 + \alpha\bar{x}| - c(\bar{y},\bar{t})|p_1 - \alpha\bar{y}| \\ &\quad + g(\bar{y})H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - g(\bar{x})H_*(p_1 + \alpha\bar{x}, X + \alpha I) \\ &\quad + \langle Dg(\bar{x}) - Dg(\bar{y}), p_1\rangle + \langle Dg(\bar{y}), \alpha\bar{y}\rangle + \langle Dg(\bar{x}), \alpha\bar{x}\rangle \\ &\leqslant \alpha\big(\|c\|_{L^\infty(\mathbb{R}^2\times[0,T))} + L_g\big)\big(|\bar{x}| + |\bar{y}|\big) + \big(c(\bar{x},\bar{t}) - c(\bar{y},\bar{t})\big)|p_1| \\ &\quad + g(\bar{y})H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - g(\bar{x})H_*(p_1 + \alpha\bar{x}, X + \alpha I) \\ &\quad + L_g|\bar{x} - \bar{y}\|p_1|. \end{aligned}$$

Let us assume that $g(\bar{y}) \leqslant g(\bar{x})$. In this case,

$$\begin{aligned} & g(\bar{y})H^*(p_1 - \alpha\bar{y}, Y - \alpha I) - g(\bar{x})H_*(p_1 + \alpha\bar{x}, X + \alpha I) \\ &= \underbrace{\big(g(\bar{x}) - g(\bar{y})\big)}_{\geqslant 0}\big(\underbrace{-H^*(p_1 - \alpha\bar{y}, Y - \alpha I)}_{\leqslant C_3}\big) + g(\bar{x})H^*(p_1 - \alpha\bar{y}, Y - \alpha I) \\ &\quad - g(\bar{x})H_*(p_1 + \alpha\bar{x}, X + \alpha I) \end{aligned}$$

$$\leqslant C_3 L_g |\bar{x}-\bar{y}| + g(\bar{x}) H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - g(\bar{x}) H_*(p_1 + \alpha \bar{x}, X + \alpha I).$$

Besides,

$$\begin{aligned}
&g(\bar{x}) H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - g(\bar{x}) H_*(p_1 + \alpha \bar{x}, X + \alpha I) \\
&\quad = g(\bar{x})\big(H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H(p_1, Y) + H(p_1, Y)\big) \\
&\qquad - g(\bar{x})\big(H_*(p_1 + \alpha \bar{x}, X + \alpha I) - H(p_1, X) + H(p_1, X)\big) \\
&\quad = g(\bar{x})\big(H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H(p_1, Y)\big) \\
&\qquad - g(\bar{x})\big(H_*(p_1 + \alpha \bar{x}, X + \alpha I) - H(p_1, X)\big) \\
&\qquad + g(\bar{x}) \underbrace{\big(H(p_1, Y) - H(p_1, X)\big)}_{\leqslant 0 \text{ since } X \leqslant Y \text{ from the matrix inequality}}.
\end{aligned}$$

In the case where $g(\bar{y}) \geqslant g(\bar{x})$, we obtain the same result using the inequality $H_*(p_1 + \alpha \bar{x}, X + \alpha I) \leqslant C_3$. We then have:

$$\begin{aligned}
&b'(\bar{t})(\bar{x_2} - \bar{y_2}) + \frac{\gamma}{T^2} \\
&\quad \leqslant \alpha\big(\|c\|_{L^\infty(\mathbb{R}^2 \times [0,T))} + L_g\big)\big(|\bar{x}| + |\bar{y}|\big) + C_3 L_g\big(|\bar{x_1} - \bar{y}_1| + \bar{y}_2 - \bar{x}_2\big) \\
&\qquad + (L_c + L_g)\left(\frac{|\bar{x_1} - \bar{y_1}|^2}{\epsilon} + b(\bar{t})|\bar{x_1} - \bar{y_1}| + \frac{|\bar{x_1} - \bar{y_1}|}{\epsilon}(\bar{y_2} - \bar{x_2}) + b(\bar{t})(\bar{y_2} - \bar{x_2})\right) \\
&\qquad + g(\bar{x})\big[H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H^*(p_1, Y)\big] \\
&\qquad + g(\bar{x})\big[H_*(p_1, X) - H_*(p_1 + \alpha \bar{x}, X + \alpha I)\big].
\end{aligned}$$

Moreover, since u is $B(t)$-Lipschitz continuous in space, we have $|p_1| \leqslant B(\bar{t}) + \alpha|\bar{y}|$, hence $|\bar{x_1} - \bar{y_1}| \leqslant B(\bar{t})\epsilon + \alpha\epsilon|\bar{y}|$. Thus,

$$\begin{aligned}
&b'(\bar{t})(\bar{x_2} - \bar{y_2}) + \frac{\gamma}{T^2} \\
&\quad \leqslant \alpha\big(\|c\|_{L^\infty(\mathbb{R}^2 \times [0,T))} + L_g\big)\big(|\bar{x}| + |\bar{y}|\big) + C_3 L_g\big(B(T)\epsilon + \alpha\epsilon|\bar{y}| + \bar{y}_2 - \bar{x}_2\big) \\
&\qquad + (L_c + L_g)\left(\frac{(B(T)\epsilon + \alpha\epsilon|\bar{y}|)^2}{\epsilon} + b_0 B(T)\epsilon + b_0\alpha\epsilon|\bar{y}| + B(\bar{t})(\bar{y_2} - \bar{x_2})\right. \\
&\qquad \left. + \alpha|\bar{y}|(\bar{y_2} - \bar{x_2}) + b(\bar{t})(\bar{y_2} - \bar{x_2})\right) \\
&\qquad + g(\bar{x})\big[H^*(p_1 - \alpha \bar{y}, Y - \alpha I) - H^*(p_1, Y)\big] \\
&\qquad + g(\bar{x})\big[H_*(p_1, X) - H_*(p_1 + \alpha \bar{x}, X + \alpha I)\big].
\end{aligned}$$

But $\bar{M} > 0$ so

$$\begin{aligned}
&\frac{|\bar{x_1} - \bar{y_1}|^2}{2\epsilon} + \frac{\alpha}{2}\big(|\bar{x}|^2 + |\bar{y}|^2\big) \\
&\quad \leqslant u(\bar{x_1}, \bar{x_2}, \bar{t}) - u(\bar{y_1}, \bar{y_2}, \bar{t}) - b(\bar{t})(\bar{x_2} - \bar{y_2})
\end{aligned}$$

$$
\begin{aligned}
&\leqslant u_0(\bar{x_1},\bar{x_2}) - u_0(\bar{y_1},\bar{y_2}) + 2C_1T + b_0(\bar{y_2}-\bar{x_2}) \\
&\leqslant 2C_1T + b_0(\bar{y_2}-\bar{x_2}) + \big(u_0(\bar{x_1},\bar{x_2}) - u_0(\bar{x_1},\bar{y_2}) + u_0(\bar{x_1},\bar{y_2}) - u_0(\bar{y_1},\bar{y_2})\big).
\end{aligned}
$$

Moreover, $u_0(\bar{x_1},\bar{x_2}) - u_0(\bar{x_1},\bar{y_2}) + b_0(\bar{y_2}-\bar{x_2}) \leqslant 0$ and $|u_0(\bar{x_1},\bar{y_2}) - u_0(\bar{y_1},\bar{y_2})| \leqslant B_0|\bar{x_1}-\bar{y_1}|$. Consequently,

$$
\frac{\alpha}{2}\big(|\bar{x}|^2+|\bar{y}|^2\big) \leqslant 2C_1T + B_0|\bar{x_1}-\bar{y_1}| - \frac{|\bar{x_1}-\bar{y_1}|^2}{2\epsilon} \leqslant 2C_1T + \frac{B_0^2\epsilon}{2} \leqslant 2C_1T + \frac{B_0^2}{2},
$$

for ϵ small enough. Thus $\lim_{\alpha\to 0}\alpha\bar{x} = \lim_{\alpha\to 0}\alpha\bar{y} = 0$ and we can assume, for α small enough that $\alpha|\bar{x}| \leqslant 1$ and $\alpha|\bar{y}| \leqslant 1$. With this result in mind, taking ϵ sufficiently small, it yields:

$$
\begin{aligned}
b'(\bar{t})(\bar{x_2}-\bar{y_2}) + \frac{\gamma}{2T^2} \leqslant{} & \alpha\big(\|c\|_{L^\infty(\mathbb{R}^2\times[0,T))} + L_g\big)\big(|\bar{x}|+|\bar{y}|\big) + C_3L_g(\bar{y}_2-\bar{x}_2) \\
&+ (L_c+L_g)\big(2B(\bar{t})+1\big)(\bar{y_2}-\bar{x_2}) \\
&+ g(\bar{x})\big[H^*(p_1-\alpha\bar{y}, Y-\alpha I) - H^*(p_1,Y)\big] \\
&+ g(\bar{x})\big[H_*(p_1,X) - H_*(p_1+\alpha\bar{x}, X+\alpha I)\big].
\end{aligned}
$$

But $b'(\bar{t}) = -(2B(\bar{t})+1)(L_c+L_g) - C_3L_g$ so,

$$
\begin{aligned}
\frac{\gamma}{2T^2} \leqslant{} & \alpha\big(\|c\|_{L^\infty(\mathbb{R}^2\times[0,T))} + L_g\big)\big(|\bar{x}|+|\bar{y}|\big) \\
&+ g(\bar{x})\big[H^*(p_1-\alpha\bar{y}, Y-\alpha I) - H^*(p_1,Y)\big] \\
&+ g(\bar{x})\big[H_*(p_1,X) - H_*(p_1+\alpha\bar{x}, X+\alpha I)\big]. \qquad (3.4)
\end{aligned}
$$

Remark that X and Y are bounded independently of α from the matrix inequality. This is also the case for p_1. So there exits $\alpha_n \to 0$ such that $\bar{t} \to t_\infty$, $p_1 \to p_\infty$ and $(X,Y) \to (X_\infty, Y_\infty)$. Sending α_n to 0 in (3.4) and using the fact that $\lim_{\alpha\to 0}\alpha\bar{x} = \lim_{\alpha\to 0}\alpha\bar{y} = 0$, $p_1 \neq 0$ and $p_\infty \neq 0$, it yields:

$$
\frac{\gamma}{2T^2} \leqslant 0,
$$

which is absurd. □

4. The nonlocal problem

The space $BV(\mathbb{R}^2)$ is the space of bounded variation functions. We denote by $|\cdot|_{BV}$ the BV-norm. Let us define by $L^1_{unif}(\mathbb{R}^2)$ the space:

$$
L^1_{unif}\big(\mathbb{R}^2\big) = \big\{f:\mathbb{R}^2\to\mathbb{R},\ \|f\|_{L^1_{unif}(\mathbb{R}^2)} < \infty\big\},
$$

with

$$
\|f\|_{L^1_{unif}(\mathbb{R}^2)} = \sup_{x\in\mathbb{R}^2}\int\limits_{Q(x)} |f|,
$$

with $Q(x)$ the unit square centered at x: $Q(x) = \{x' \in \mathbb{R}^2,\ |x_i - x'_i| \leqslant \frac{1}{2}\}$, and by $L^\infty_{int}(\mathbb{R}^2)$ the space:

$$L^\infty_{int}(\mathbb{R}^2) = \{f : \mathbb{R}^2 \to \mathbb{R},\ \|f\|_{L^\infty_{int}(\mathbb{R}^2)} < \infty\},$$

with $\|f\|_{L^\infty_{int}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \|f\|_{L^\infty(Q(x))}$.

Theorem 6 *(Short time existence and uniqueness). Assume* (H1)–(H2) *and let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be such that $Du_0 \in W^{1,\infty}(\mathbb{R}^2)$ and*:

$$|Du_0| < B_0 \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \frac{\partial u_0}{\partial x_2} > b_0 > 0 \quad \text{in } \mathbb{R}^2.$$

Let c_0 satisfies $c_0 \in L^\infty_{int}(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$. Then there exists a unique viscosity solution of problem (2.2) *in $\mathbb{R}^2 \times [0, T^*)$ with*:

$$T^* = \inf\left(\frac{\ln\left(\frac{b_0 C_2}{8B_0(|c_0|_{BV(\mathbb{R}^2)} + L_g)} + 1\right)}{C_2}, \frac{b_0}{4C_4}, \frac{b_0}{16B_0\|c_0\|_{L^\infty_{int}(\mathbb{R}^2)}}, \frac{\ln 2}{C_2}\right),$$

where

$$C_4 = |c_0|_{BV(\mathbb{R}^2)} + L_g + \frac{L_g}{\delta}\left(2C_1 + 2\|c_0\|_{L^1} B_0 + 2L_g B_0\right) \quad \text{and} \quad C_2 = |c_0|_{BV} + 5L_g^2 + L_g.$$

Moreover, the solution satisfies:

$$\left|Du(x,t)\right| \leqslant 2B_0 \quad \text{on } \mathbb{R}^2 \times [0, T^*), \qquad \frac{\partial u}{\partial x_2}(x,t) > \frac{b_0}{2} > 0 \quad \text{on } \mathbb{R}^2 \times [0, T^*),$$
$$\left|u_t(x,t)\right| \leqslant 2C_1 \quad \text{on } \mathbb{R}^2 \times [0, T^*).$$

We need the three following lemmas.

Lemma 2 *(Estimate on the characteristic functions). Let $u^1 \in \mathcal{C}(\mathbb{R}^2)$ satisfying*

$$\frac{\partial u^1}{\partial x_2} \geqslant b$$

in the distribution sense for some $b > 0$ and $u^2 \in L^\infty_{loc}(\mathbb{R}^2)$ satisfying the same condition. Then we have the following estimate:

$$\left\|[u^2] - [u^1]\right\|_{L^1_{unif}} \leqslant \frac{2}{b}\left\|u^2 - u^1\right\|_{L^\infty}.$$

Lemma 3 *(Convolution inequality). For every $f \in L^1_{unif}(\mathbb{R}^2)$ and $g \in L^\infty_{int}(\mathbb{R}^2)$, the convolution product $f * g$ is bounded and satisfies*:

$$\|f * g\|_{L^\infty(\mathbb{R}^2)} \leqslant \|f\|_{L^1_{unif}(\mathbb{R}^2)}\|g\|_{L^\infty_{int}(\mathbb{R}^2)}.$$

A proof of these two lemmas can be found respectively in [2] and [3].

Lemma 4 *(Stability of the solution with respect to the velocity). Let $T > 0$. We consider for $i = 1, 2$ two different equations*:

$$\begin{cases} u^i_t = c^i(x,t)\left|Du^i\right| - F\left(x, Du^i, D^2u^i\right) & \text{in } \mathbb{R}^2 \times (0,T), \\ u^i(x,0) = u_0(x), \end{cases} \tag{4.5}$$

c^i, u_0 and F satisfying the previous assumptions. Then for every $t \in [0,T)$, we have:

$$\left\|u^1(\cdot,t) - u^2(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \left\|c^1 - c^2\right\|_{L^\infty(\mathbb{R}^2\times(0,T))} \int_0^T B(s)\,ds,$$

where u^i are the solutions of (4.5), *$B(t) = B_0 e^{(L_c+5L_g^2+L_g)t}$ with $L_c = \sup_i L_{c^i}$.*

Proof. We set $K = \|c^1 - c^2\|_{L^\infty(\mathbb{R}^2\times(0,T))}$. We remark that u^1 is a sub-solution of

$$u_t - c^2(x,t)|Du| + F\left(x, Du, D^2u\right) - KB(t) = 0.$$

Indeed, we have:

$$\begin{aligned} &u^1_t - c^2(x,t)\left|Du^1\right| + F\left(x, Du^1, D^2u^1\right) \\ &\quad = c^1(x,t)\left|Du^1\right| - F\left(x, Du^1, D^2u^1\right) - c^2(x,t)\left|Du^1\right| + F\left(x, Du^1, D^2u^1\right) \\ &\quad \leqslant \left\|c^1 - c^2\right\|_{L^\infty(\mathbb{R}^2\times(0,T))} B(t) \\ &\quad \leqslant KB(t). \end{aligned}$$

This differential inequality holds in the viscosity sense. Moreover, the function $u^2 + K\int_0^t B(s)\,ds$ is solution of the same problem. By the comparison principle, we deduce that:

$$u^1 \leqslant u^2 + K\int_0^t B(s)\,ds.$$

Switching the role of u^1 and u^2, it yields:

$$\left\|u^1(\cdot,t) - u^2(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant \left\|c^1 - c^2\right\|_{L^\infty(\mathbb{R}^2\times(0,T))} \int_0^T B(s)\,ds. \qquad \square$$

It now brings us to the proof of Theorem 6.

Proof of Theorem 6. We define the space

$$E = \left\{ u \in L^\infty_{loc}\left(\mathbb{R}^2 \times \left[0, T^*\right)\right) \text{ s.t. } \left|\ \begin{array}{l} |Du(x,t)| \leqslant 2B_0 \\ \frac{\partial u}{\partial x_2}(x,t) \geqslant \frac{b_0}{2} \\ |u_t(x,t)| \leqslant 2C_1 \end{array} \right. \right\},$$

where C_1 is defined in Proposition 1.

For $u \in E$, we set $c(x,t) = (c_0 * [u(\cdot,t)])(x)$. This function is bounded, Lipschitz continuous in space (with $L_c = |c_0|_{BV}$) and time (with $L_{c_t} = \frac{8C_1\|c_0\|_{L^\infty_{int}}}{b_0}$). Indeed,

$$\begin{aligned}\|c\|_{L^\infty(\mathbb{R}^2\times[0,T^*))} &\leqslant \sup_{t\in\mathbb{R}} \|c_0\|_{L^1(\mathbb{R}^2)} \big\|\big[u(\cdot,t)\big]\big\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant \|c_0\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

Moreover, for every t,

$$\begin{aligned}\big\|Dc(\cdot,t)\big\|_{L^\infty(\mathbb{R}^2)} &= \big\|Dc_0 * \big[u(\cdot,t)\big]\big\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant |c_0|_{BV(\mathbb{R}^2)} \big\|\big[u(\cdot,t)\big]\big\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant |c_0|_{BV(\mathbb{R}^2)}.\end{aligned}$$

Finally, for $0 < t, s < T^*$:

$$\begin{aligned}\big|c(x,t) - c(x,s)\big| &= \big|c_0 * \big[u(\cdot,t)\big](x) - c_0 * \big[u(\cdot,s)\big](x)\big| \\ &= \big|c_0 * \big(\big[u(\cdot,t)\big] - \big[u(\cdot,s)\big]\big)(x)\big| \\ &\leqslant \|c_0\|_{L^\infty_{int}(\mathbb{R}^2)} \big\|\big[u(\cdot,t)\big] - \big[u(\cdot,s)\big]\big\|_{L^1_{unif}(\mathbb{R}^2)} \\ &\leqslant \frac{4\|c_0\|_{L^\infty_{int}(\mathbb{R}^2)}}{b_0} \big\|u(\cdot,t) - u(\cdot,s)\big\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant \frac{8C_1\|c_0\|_{L^\infty_{int}(\mathbb{R}^2)}}{b_0} |t-s|.\end{aligned}$$

For $u \in E$, we then define $v = \Phi(u)$ as the unique viscosity solution of:

$$\begin{cases} v_t = \big(c_0 * [u]\big)|Dv| - F\big(x, Dv, D^2 v\big) & \text{in } \mathbb{R}^2 \times \big(0, T^*\big), \\ v(x, t=0) = u_0(x) & \text{in } \mathbb{R}^2. \end{cases}$$

We show that $\Phi : E \to E$ is a contraction. First, we show that Φ is well defined. We have:

$$\big\|Dv(\cdot,t)\big\| \leqslant B(t) \leqslant B_0 e^{(L_c + 5L_g^2 + L_g)T^*} \leqslant 2B_0,$$

by definition of T^*.

Moreover, by Proposition 2, v is Lipschitz continuous in time and satisfies

$$\begin{aligned}\|v_t\|_{L^\infty} &\leqslant C_1 + L_{c_t} T^* B\big(T^*\big) \leqslant C_1 + 2L_{c_t} B_0 T^* \\ &\leqslant C_1\left(1 + \frac{16B_0\|c_0\|_{L^\infty_{int}(\mathbb{R}^2)}}{b_0} T^*\right) \leqslant 2C_1\end{aligned}$$

by definition of T^*. Finally, by Theorem 5, we have

$$\frac{\partial v}{\partial x_2} \geqslant b(t) \geqslant b_0 - 2\big(|c_0|_{BV(\mathbb{R}^2)} + L_g\big)\frac{B_0}{C_2}\big(e^{C_2 t} - 1\big) - C_4 t,$$

where $C_2 = L_c + 5L_g^2 + L_g$ and $C_4 = |c_0|_{BV(\mathbb{R}^2)} + L_g + \frac{L_g}{\delta}(2C_1 + 2\|c_0\|_{L^1(\mathbb{R}^2)} B_0 + 2L_g B_0)$. To ensure that $\frac{\partial v}{\partial x_2} \geqslant \frac{b_0}{2}$, it suffices to have that:

$$C_4 T^* \leqslant \frac{b_0}{4}$$

and

$$2\big(|c_0|_{BV(\mathbb{R}^2)} + L_g\big)\frac{B_0}{C_2}\big(e^{C_2 T^*} - 1\big) \leqslant \frac{b_0}{4}.$$

These two inequalities are true owing to the choice of T^*.

It thus remains to be shown that Φ is a contraction. For $v^i = \Phi(u^i)$, according to Lemmas 3 and 4, we have:

$$\begin{aligned}
\|v^2 - v^1\|_{L^\infty(\mathbb{R}^2\times(0,T^*))} &\leqslant 2B_0 T^* \|c_0 * [u^2] - c_0 * [u^1]\|_{L^\infty(\mathbb{R}^2\times(0,T^*))} \\
&\leqslant 2B_0 T^* \|c_0\|_{L^\infty_{int}(\mathbb{R}^2)} \sup_{t\in(0,T^*)} \big\|[u^2(\cdot,t)] - [u^1(\cdot,t)]\big\|_{L^1_{unif}} \\
&\leqslant \frac{8B_0 T^*}{b_0}\|c_0\|_{L^\infty_{int}(\mathbb{R}^2)} \|u^2 - u^1\|_{L^\infty(\mathbb{R}^2\times(0,T^*))} \\
&\leqslant \frac{1}{2}\|u^2 - u^1\|_{L^\infty(\mathbb{R}^2\times(0,T^*))}.
\end{aligned}$$

In conclusion, Φ is a contraction on E which is a closed set for the L^∞-topology. So there exists a unique viscosity solution to the problem in E on $(0, T^*)$. □

We are now able to prove the short time existence and uniqueness result of problem (2.2).

Proof of Theorem 1. We recall that

$$c_0 : \begin{cases} \mathbb{R}^2 \to \mathbb{R}, \\ x \mapsto \frac{4\mu}{d^2}\big(2 - \frac{2}{d^2}\|x\|_2^2\big)\exp\big(-\frac{\|x\|_2^2}{d^2}\big). \end{cases}$$

It is easy to check that $c_0 \in L^1(\mathbb{R}^2)$. It is also $\mathcal{C}^1(\mathbb{R}^2)$ so its total variation $J(c_0)$ is defined by:

$$J(c_0) = \int_{\mathbb{R}^2} |Dc_0|\, dx.$$

It is obvious that $J(c_0) < +\infty$. Consequently, $c_0 \in BV(\mathbb{R}^2)$. To finish, using inequalities, it can be proved that $c_0 \in L^\infty_{int}(\mathbb{R}^2)$. Hence we can apply Theorem 6 to get the desired result. □

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